N-Dimensional Visualization Through Raytracing

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Goal

"In mathematics, you don't understand things.
You just get used to them." —John von Neumann

I don't believe this is actually true, but practice definitely does make a difference in visualizing extra-dimensional objects.

Applications

- Pure Geometry
- Functions of Complex Variables
- Multi-dimensional Phase Spaces

Pure Geometry What does a { simplex, hypercube, 600-cell } look like?

Complex Variables A complex function of a complex variable is typically thought of as mapping a portion of the complex plane to another portion of the complex plane.

A different way of interpreting a complex function of a complex variable $f(z)$ is to consider it the set of points:

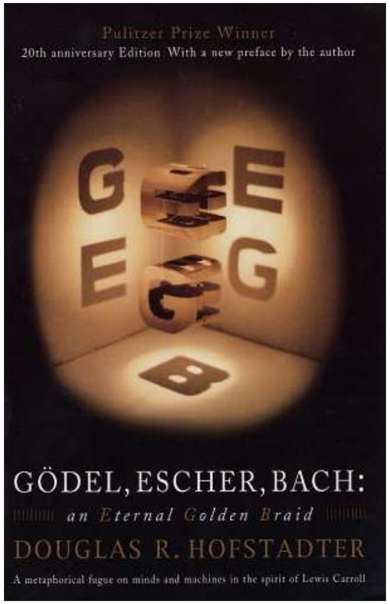
$$\{\langle a, b, c, d\rangle \,|\, f(a + ib) = c + di\}$$

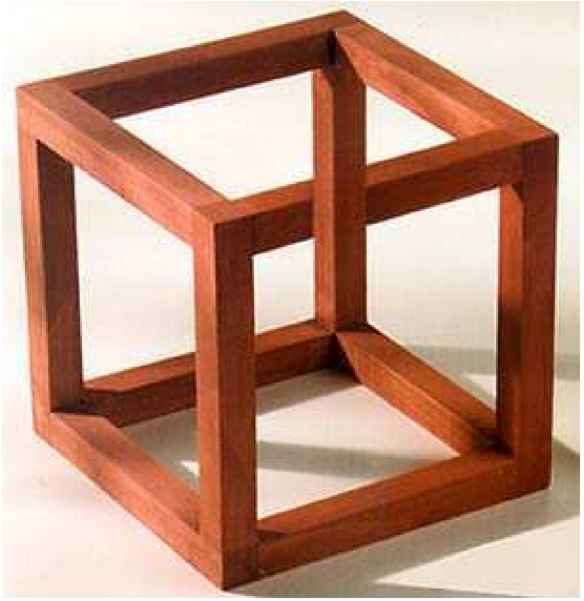
Phase Space There are many systems whose state requires more than two or three variables to describe.

- linear programs with more than three variables
- boolean equations with more than three variables
- n-item cross-correlations
- etc.

Other Applications

- Gödel-Escher-Bach Cube
- Penrose Impossible Hypercube





Truth be told, a great deal of my motivation in creating this raytracer were these play topics.

I wanted to make a 4-d picture like the cube on the cover of Gödel-Escher-Bach using more letters. I have a 5-d shape which spells out ’nklein’ when viewed from the proper axises in the proper 3-space.

John H. Conway has a cube on his desk that was a gift from Roger Penrose. The cube looks solid to us 3-d folk. But, the way the inside of the cube is structured is the 4-d analogue of the impossible cube. Sadly, I’m still trying to puzzle this one out.

Overview

- Spheres, Cylinders, and Cubes
- Intersections, Unions, and Complements
- Quadratics
- Polytopes
- Hole Cube
- Applications
    - Complex variables
    - Convex hulls
    - Sphere packings
    - Karnaugh maps
    - Phase spaces

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Cylinders Cylinders are just extrusions of lower-dimensional balls. Cubes are just special cylinders.

CSG Standard-raytracer stuff

Quadratics Quadratic surfaces

Polytopes Convex polytopes. Wythoff Constructions. Intersection of halfspaces.

Extrusions Generalized versions of cylinders

Hole Cube Pretty example.

Functions Application to real problems.

Odds and Ends E8, Monster Group, etc.

Cylinder Formulation

A cylinder can be thought of as the extrusion of an r-dimensional ball (B^r) into n-dimensional space for $0 \leq r \leq n$.

$$\sum_{i=1}^{r} x_i^2 \leq 1 \qquad \max_{i=r+1}^{n} \{x_i^2\} \leq 1$$

I'll refer to these cylinders as (r, n)-cylinders.

The extrusions are performed one at a time.

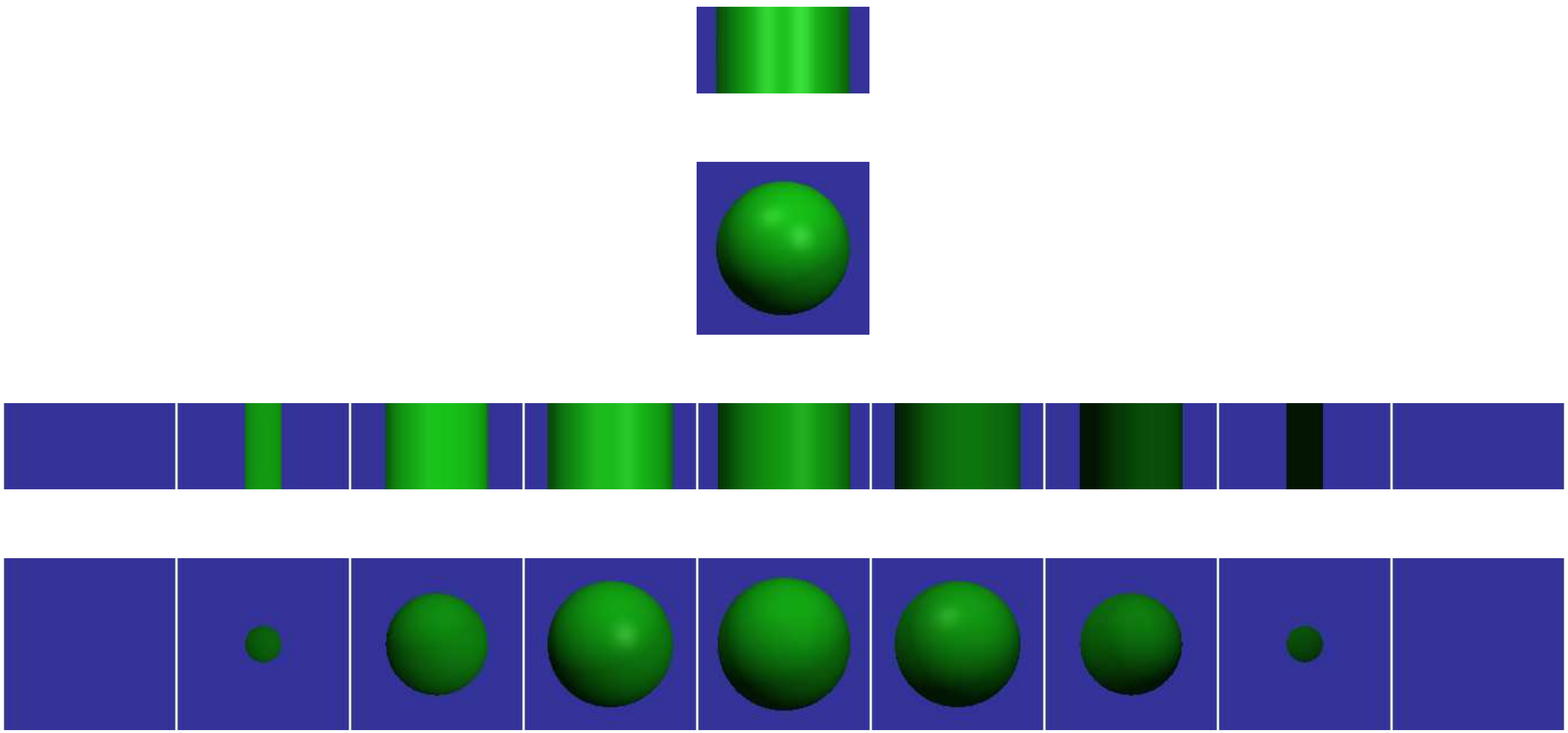
To get the duo-circle, one needs to intersect two of these.

Spheres are just the special case where $r = n$.

Cubes are just the special case where $r = 0$ (or $r = 1$).

Hyperballs

A hyperball (B^4) seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.



The hyperball is the (n, n)-cylinder.

Depicted are: $(2, 2)$-, $(3, 3)$-, $(3, 3)$-, and $(4, 4)$-cylinders.

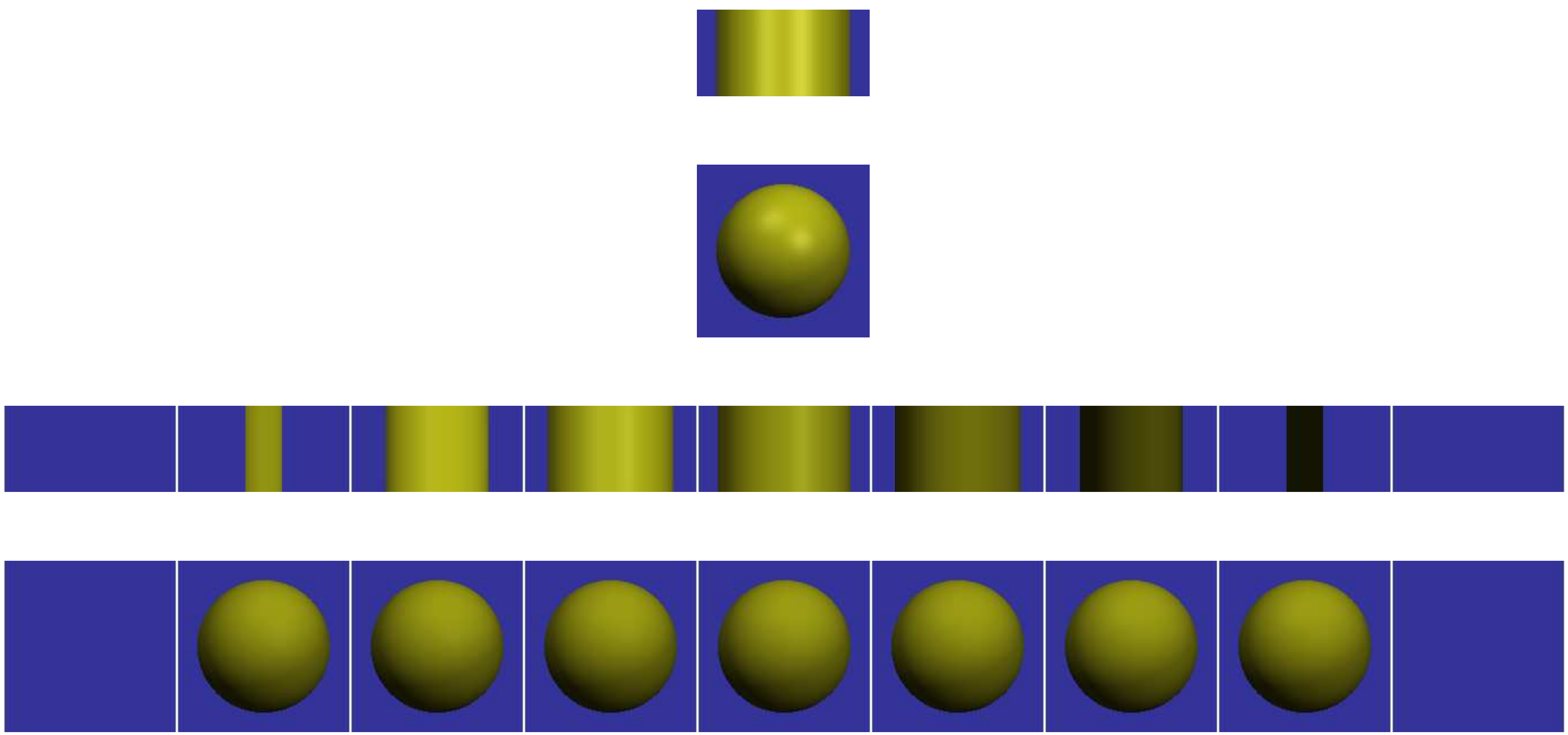
We're going to use these same basic views for lots of the pictures in this presentation. The first will be a 2-D view. Then, a 3-D view. Then, a 2-D view of the 3-D object passing through Flatland. Then, a 3-D view of the 4-D object passing through our 3-D space.

You should compare the 3-D object passing through Flatland with the 3-D object. You'll see that each of the frames in Flatland corresponds to a horizontal slice in the 3-D view right above it.

The frames in the 4-D object passing through our 3-D space bear exactly the same relation to the 4-D object that the frames in Flatland have to the 3-D object.

Spheres

A ball (B^3) extruded to 4-d and seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.

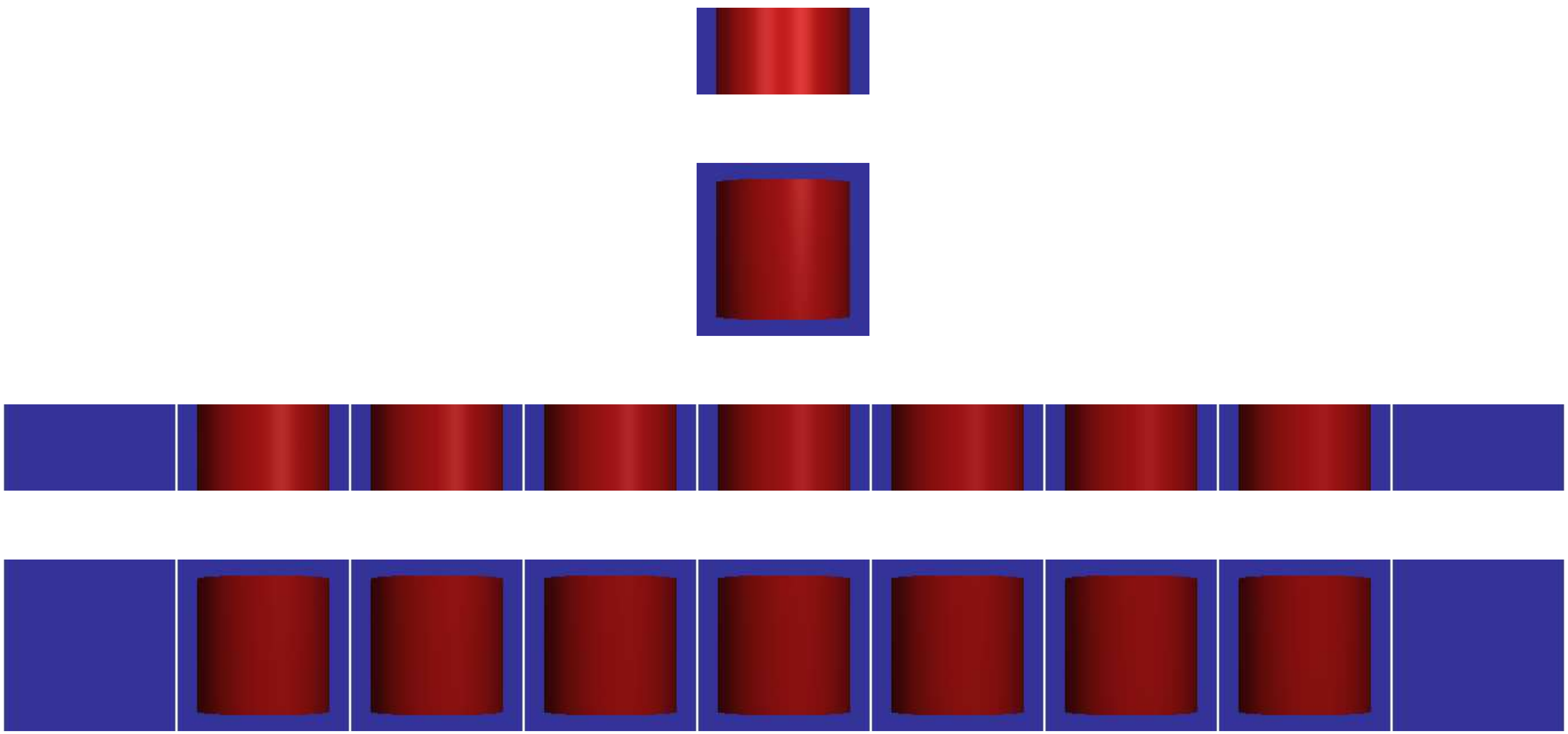


Depcited are: $(2, 2)$-, $(3, 3)$-, $(3, 3)$-, and $(3, 4)$-cylinders.

These are axis-aligned. We'll come back to this one in a minute.

Circles

A disc (B^2) extruded to 4-d and seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.

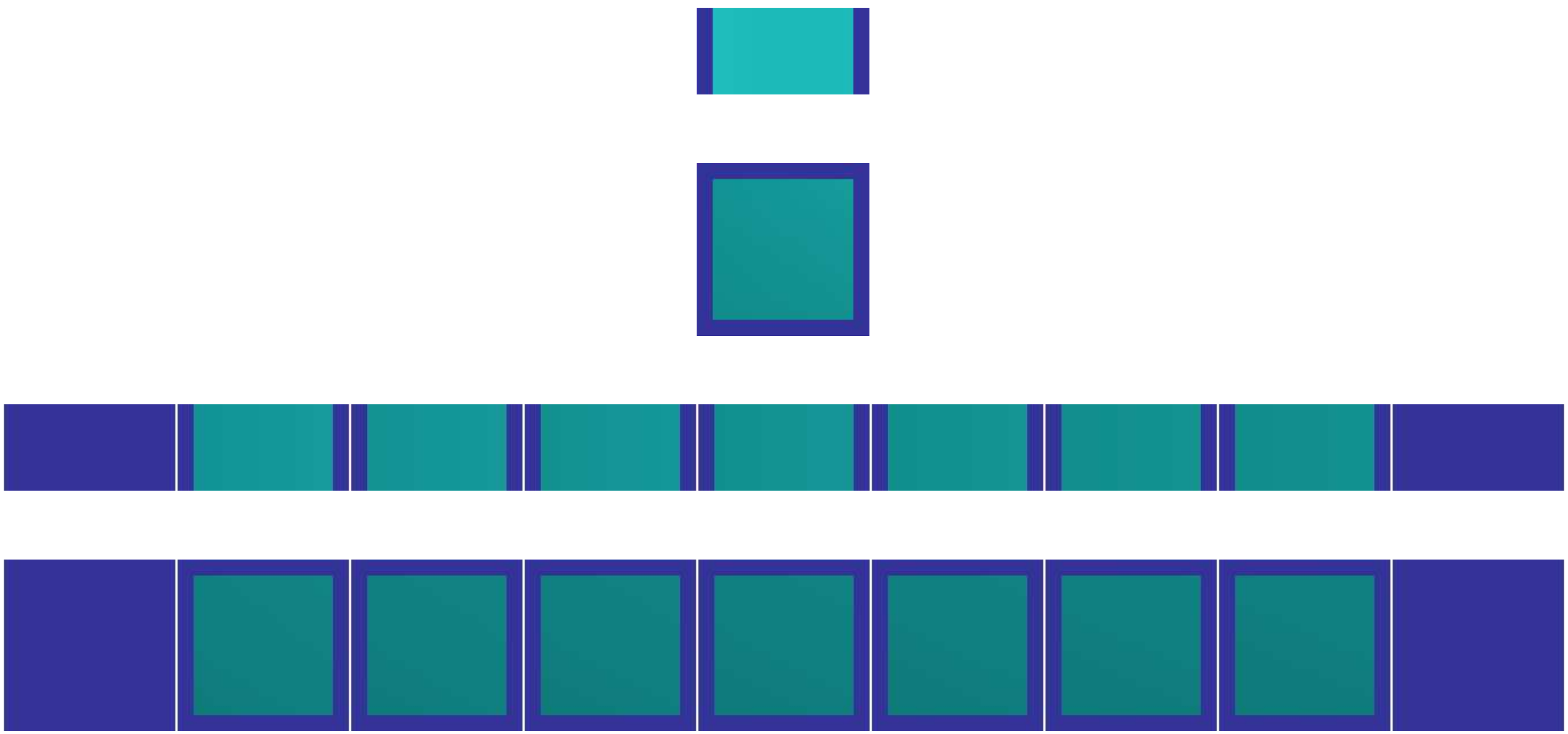


Depcited are: $(2,2)$-, $(2,3)$-, $(2,3)$-, and $(2,4)$-cylinders.

These are axis-aligned. We'll come back to this one in a minute.

Cubes

A line segment (B^1) extruded to 4-d as seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.

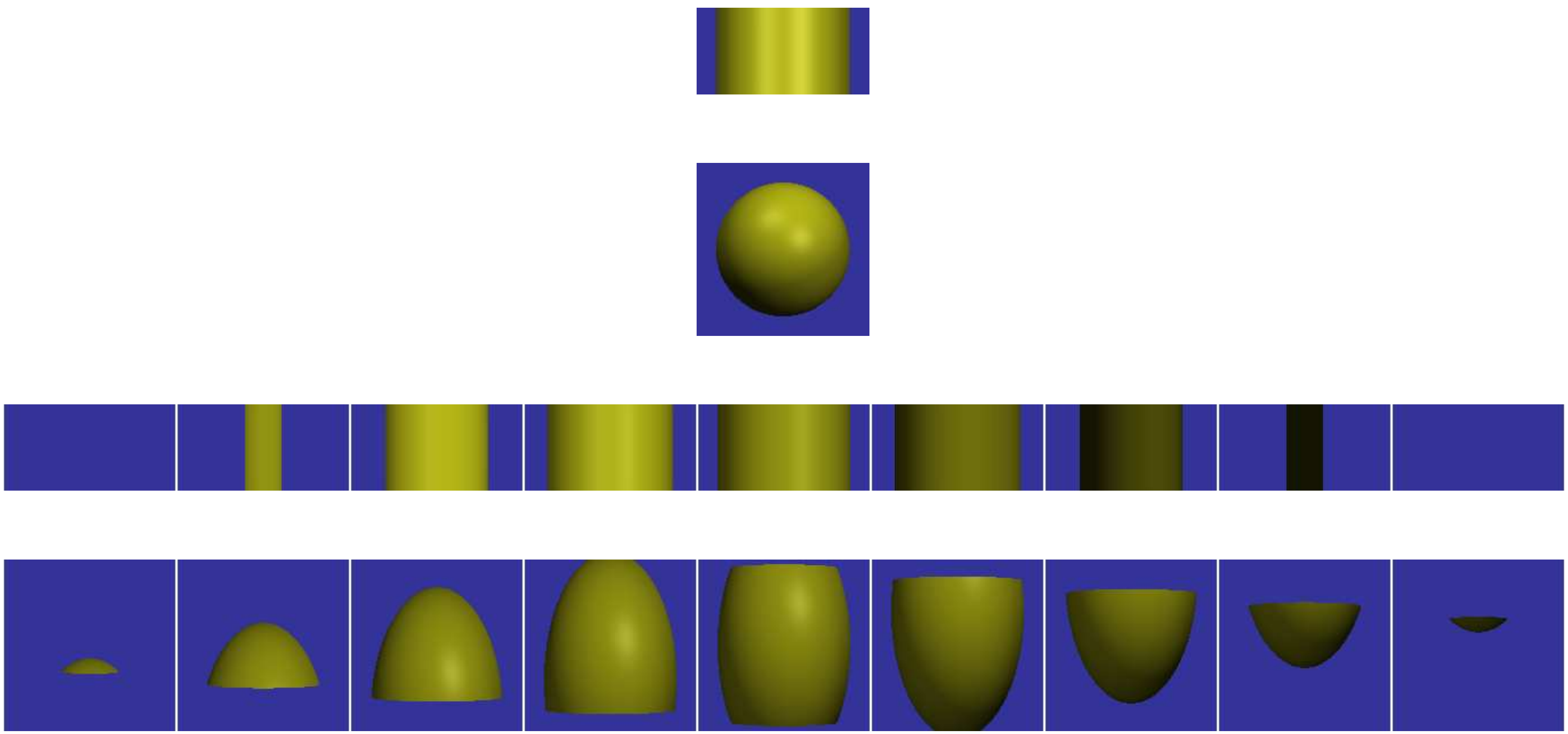


Depcited are: $(0, 2)$-, $(0, 3)$-, $(0, 3)$-, and $(0, 4)$-cylinders.

These are axis-aligned. We'll come back to this one in a minute.

Unaligned Spheres

A ball (B^3) extruded to 4-d and seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.



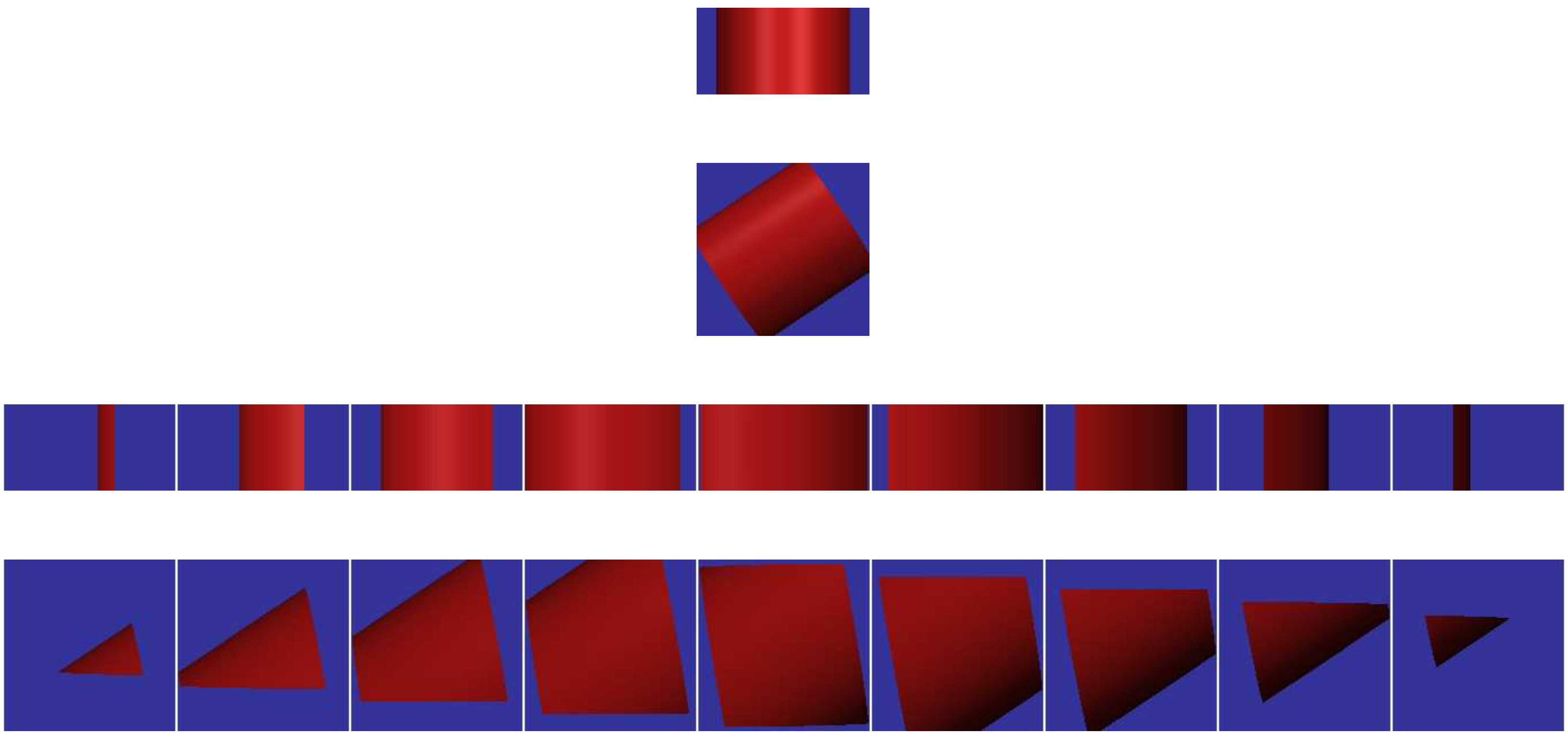
Depcited are: $(2, 2)$-, $(3, 3)$-, $(3, 3)$-, and $(3, 4)$-cylinders.

This time, they are not axis-aligned. It may take some time to understand the shape of things at the bottom. It will help to think of some of the cross-sections that one could get from an ordinary $(2, 3)$-cylinder.

In particular, draw a cord on one of the circular bases of a $(2, 3)$-cylinder. Then, cut through that cord but not parallel to the extruded axis.

Unaligned Circles

A disc (B^2) extruded to 4-d and seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.

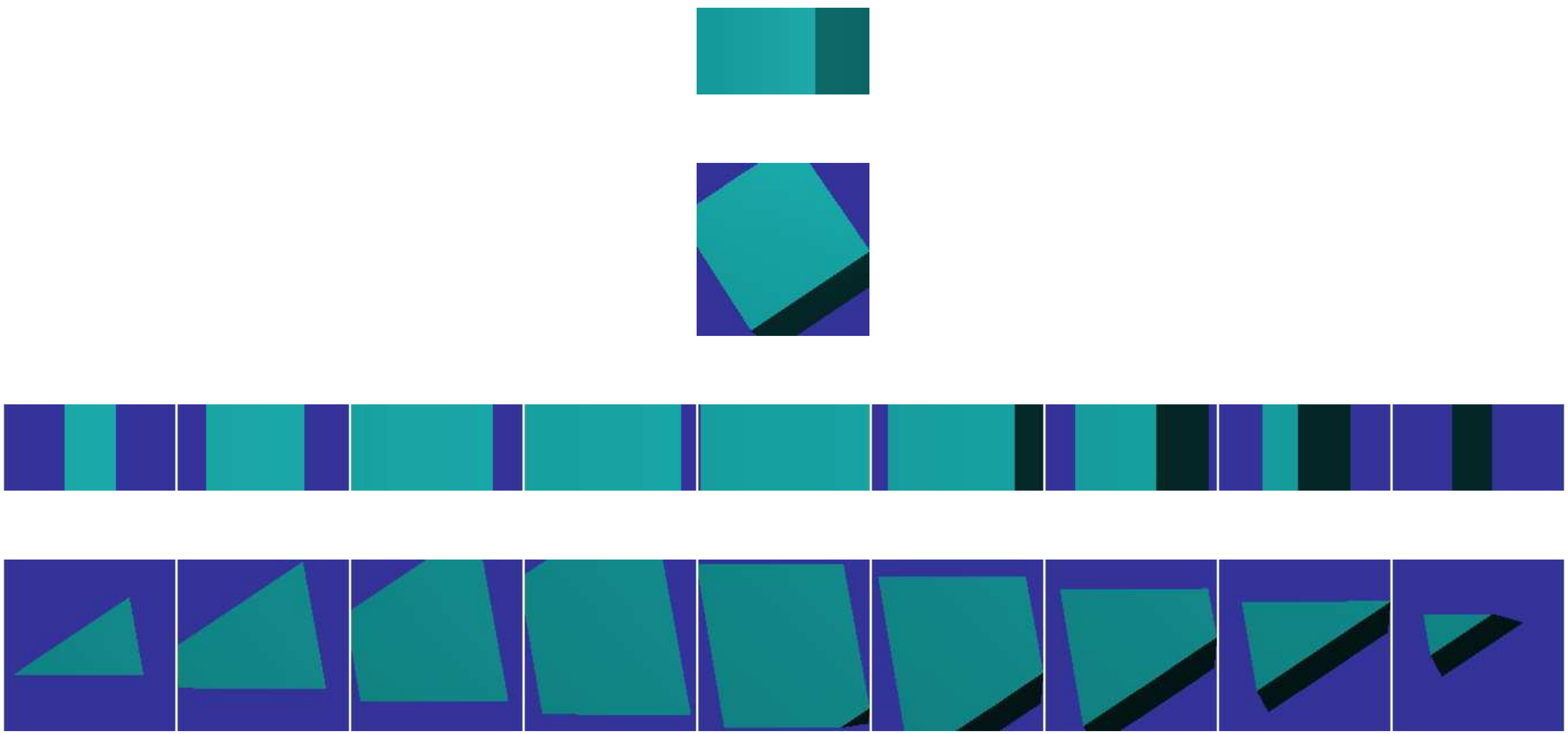


Depcited are: $(2, 2)$-, $(2, 3)$-, $(2, 3)$-, and $(2, 4)$-cylinders.

This time, they are not axis-aligned. This time, there are actually some corners not adjacent to the round parts.

Unaligned Cubes

A line segment (B^1) extruded to 4-d as seen in 2-Space, in 3-Space, passing through 2-Space, and passing through 3-Space.



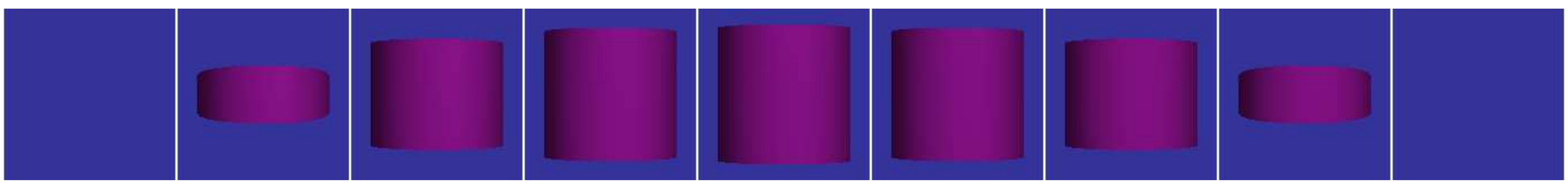
Depcited are: $(0,2)$-, $(0,3)$-, $(0,3)$-, and $(0,4)$-cylinders. This time, they are not axis-aligned.

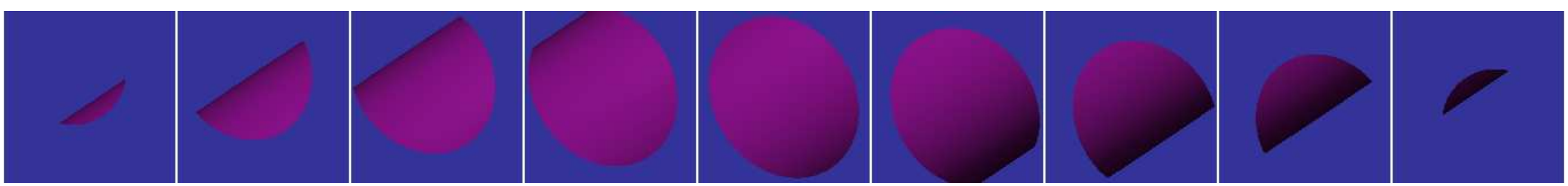
Duo-Circle

The intersection of two circles extruded into 4-d and oriented so that their “straight” sides are all orthogonal

$$x^2 + y^2 \leq 1$$

$$z^2 + w^2 \leq 1$$





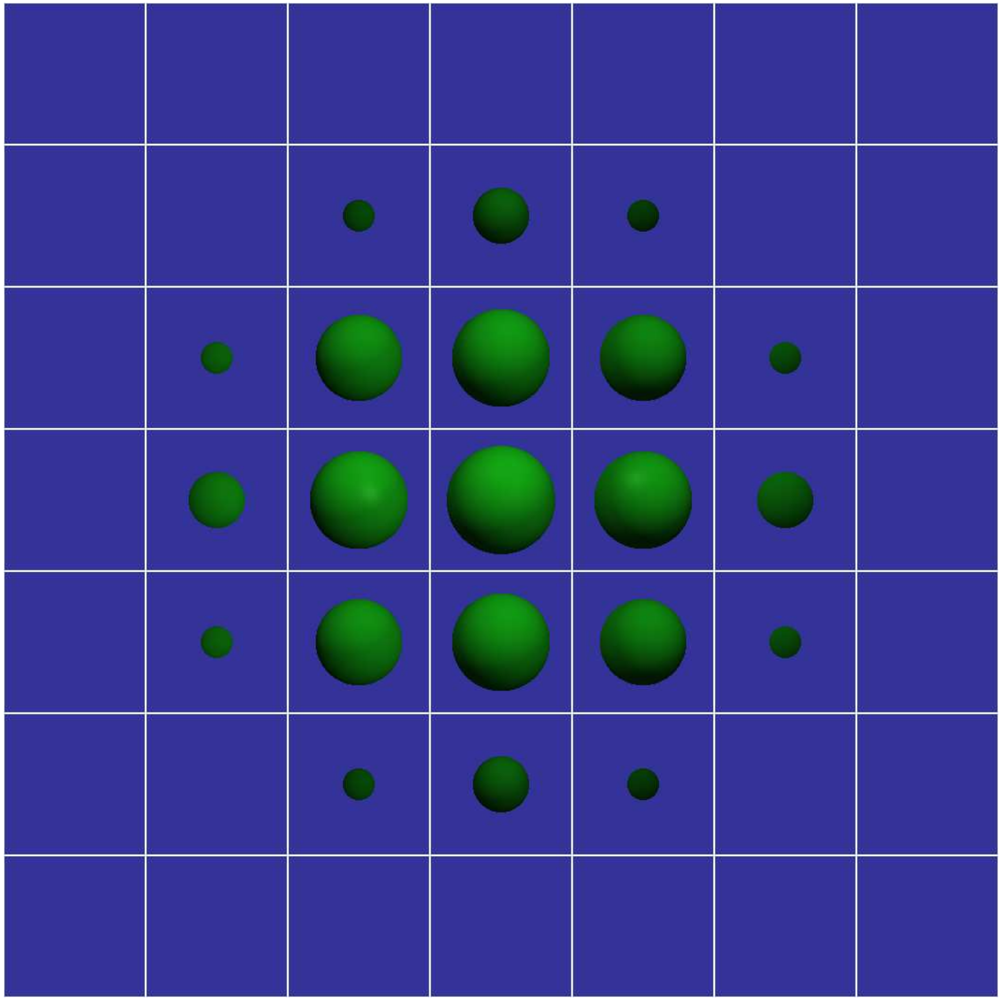
This shape has no real analogue in lower dimensions.

It is somewhat similar to the Steinmetz solid, but not really.

Compare the aligned version with the aligned spheres and aligned circles.

5-d Ball

A 5-d ball (B^5)



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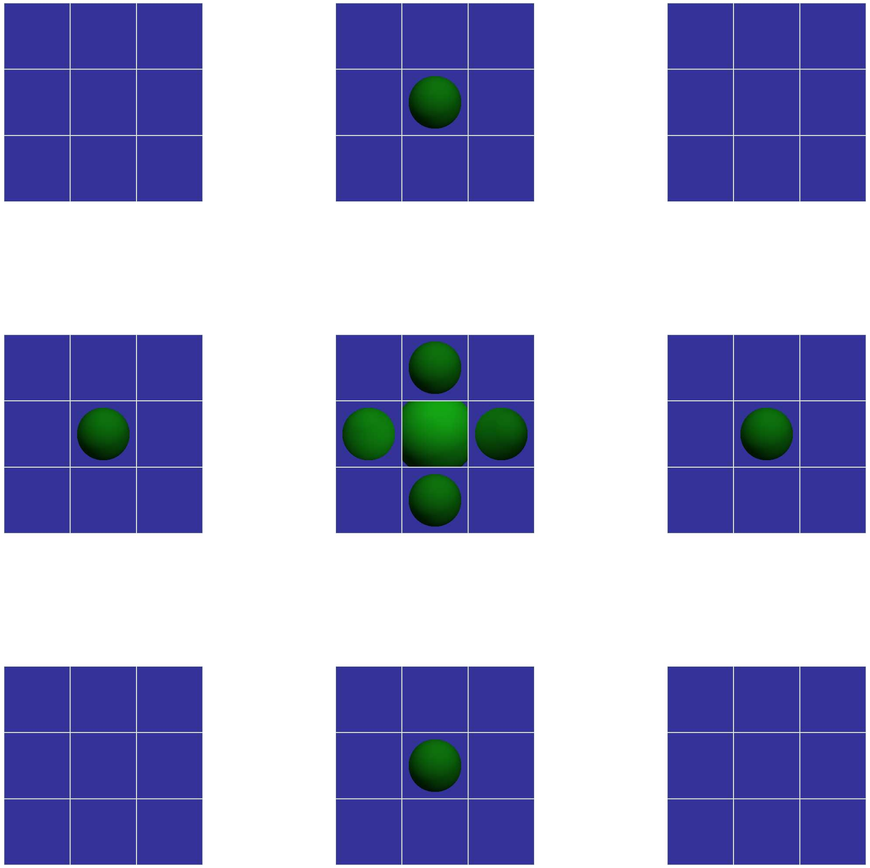
This simply takes the hyperball up in dimension.

Now the frames “progress” in two different directions.

Each horizontal (or vertical) row of frames is akin to a horizontal row in the 4-d images.

7-d Ball

A 7-d ball (B^7)



And, this goes up even further in dimension.
Each 3×3 set of frames is akin to a 5-d image.

Coordinate-Colored Hyperball

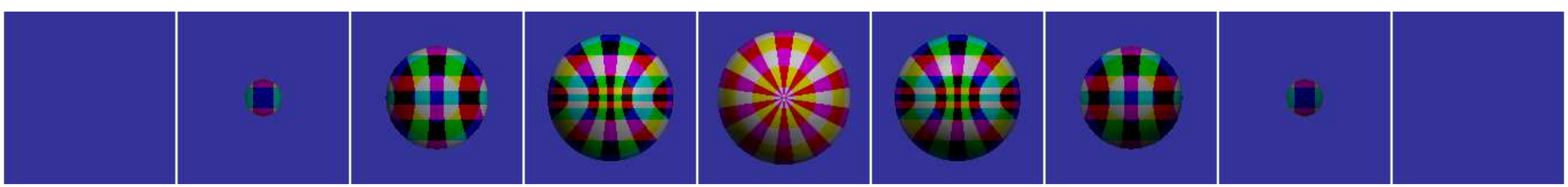
A hyperball (B^4) with its color a function of the polar coordinates

$$x = \cos\theta$$

$$y = \sin\theta\cos\phi$$

$$z = \sin\theta\sin\phi\cos\psi$$

$$w = \sin\theta\sin\phi\sin\psi$$

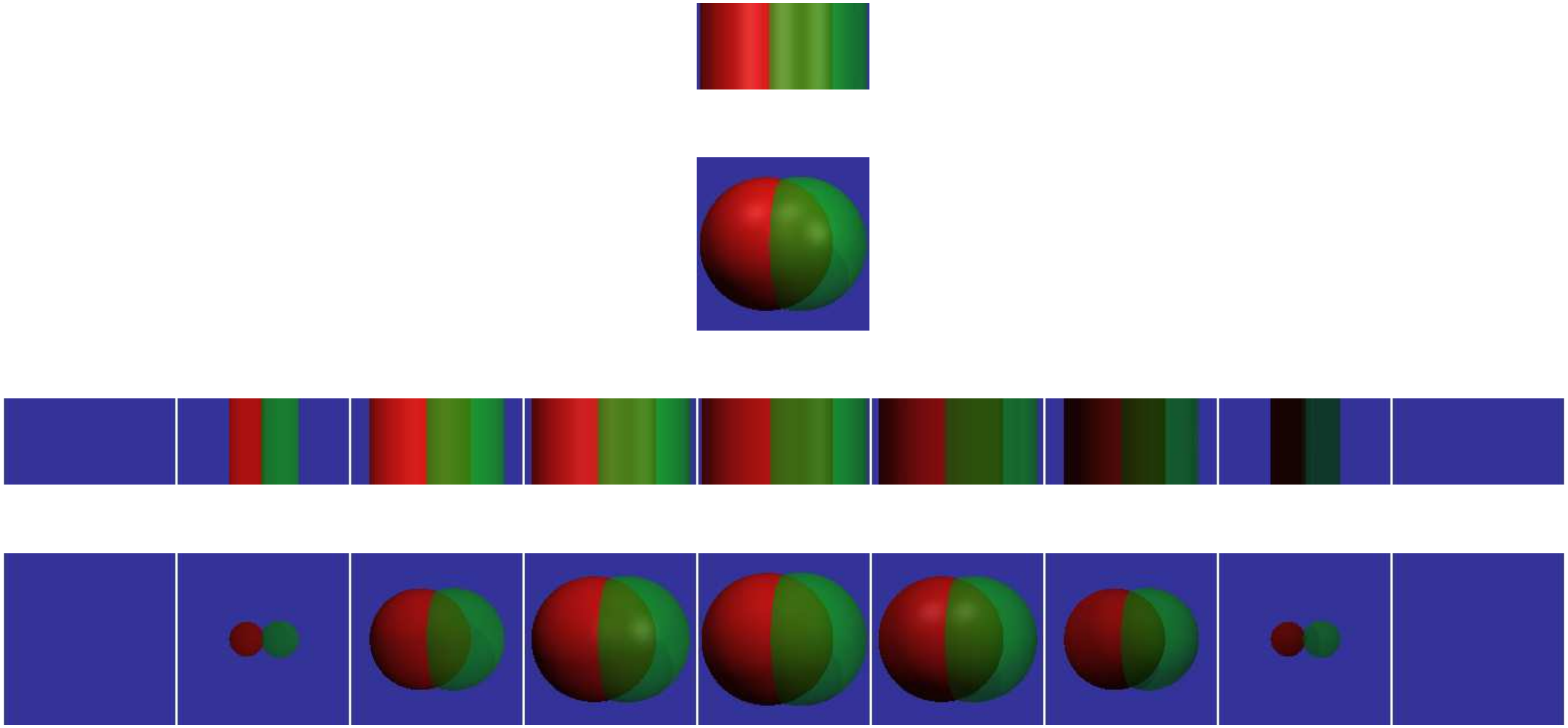


- ψ is banded in red
- ϕ is banded in green
- θ is banded in blue

Without CSG

Two overlapping balls.

$$\partial A \cup \partial B$$



CSG stands for Constructive Solid Geometry. It's a raytracing standard. The usual operations are union, intersection, and subtraction (or cut). I've implemented union, intersection, and complement.

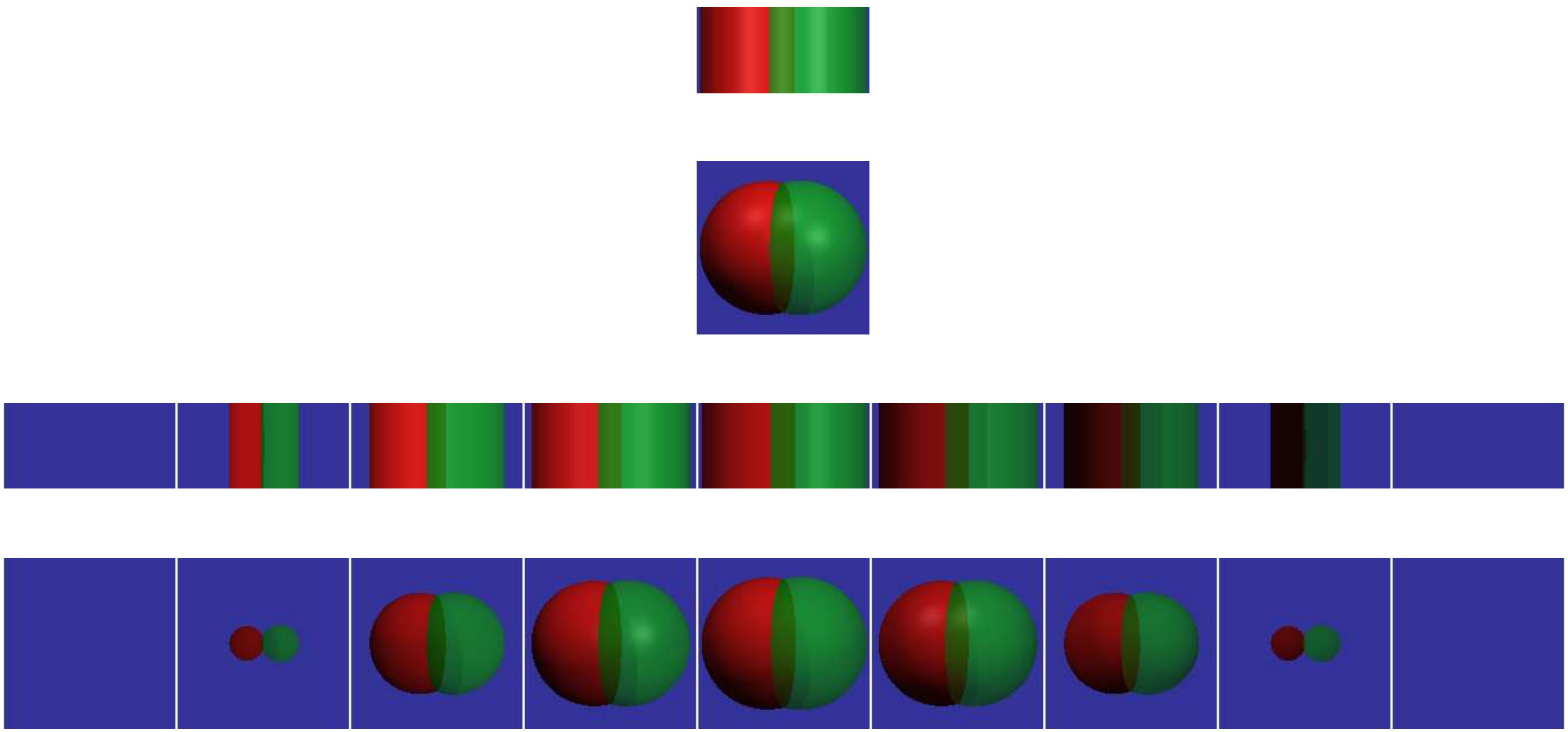
We're going to use the same pair of hyperballs (B^4) for all of the CSG pictures here.

The green hyperball will be slightly transparent so that we can see what's going on inside the balls.

Union

The union of the same two balls.

$$\partial(A \cup B)$$

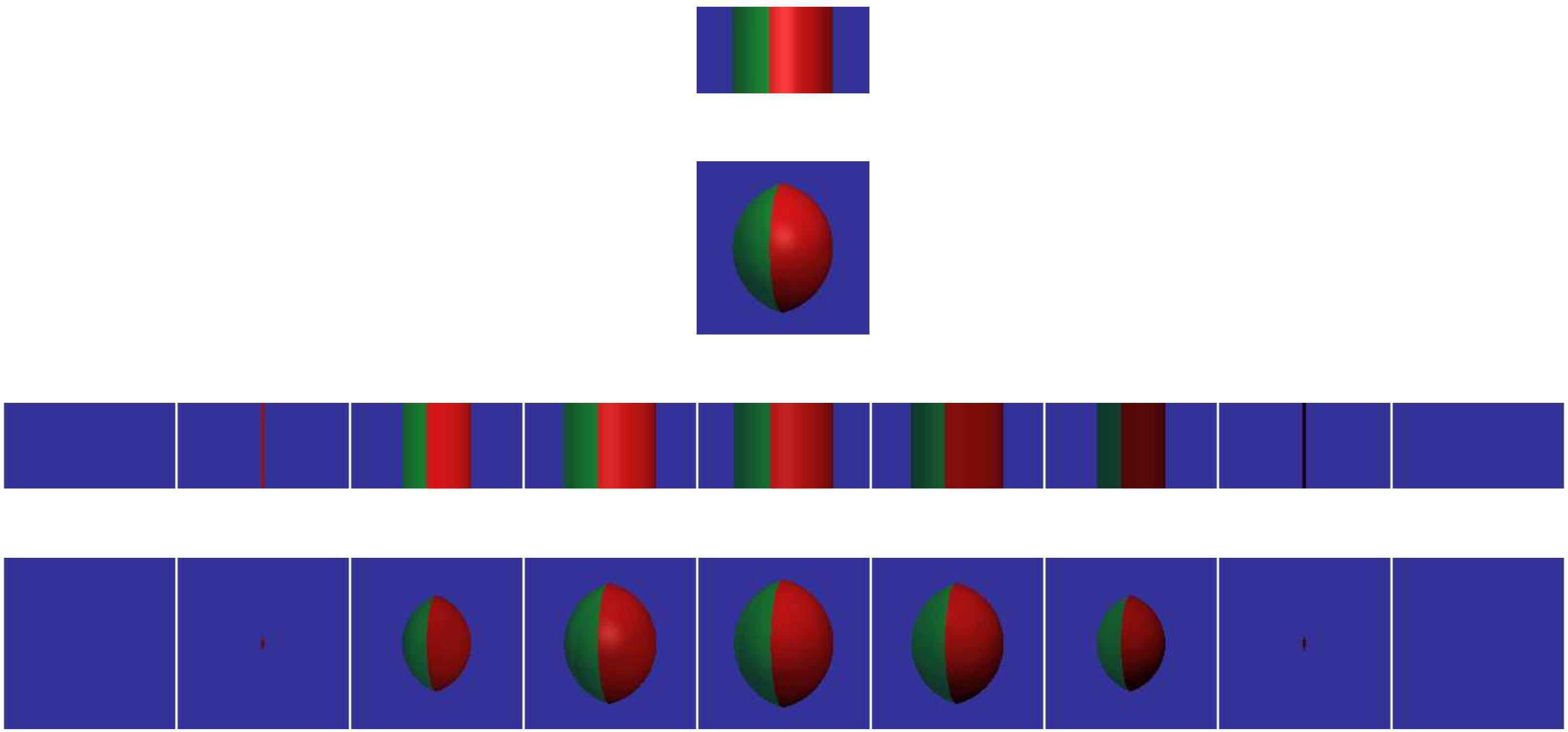


Compare what you can see through the green ball with what you could see through the green ball in the previous slide.

Intersection

The intersection of the same two balls.

$$\partial(A \cap B)$$



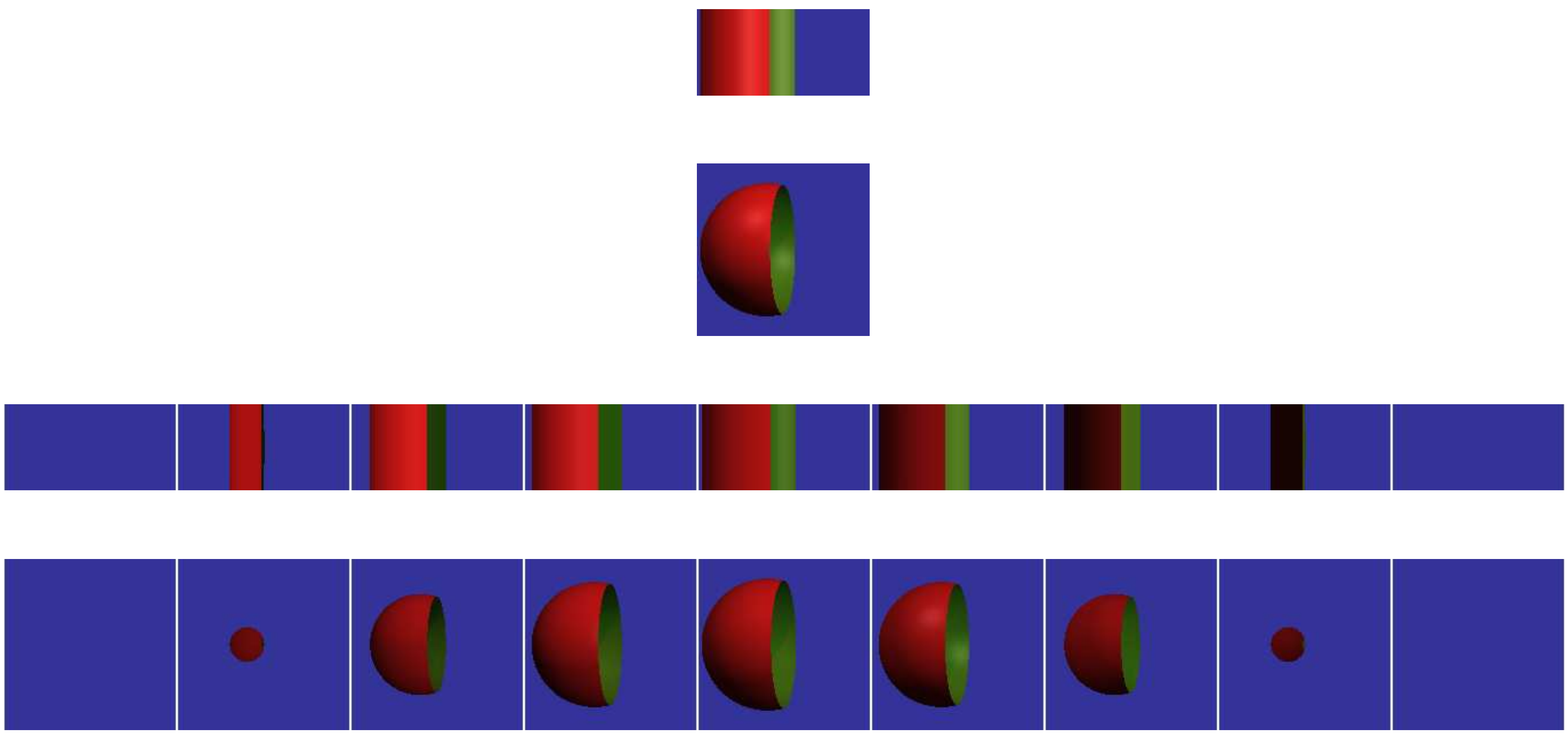
This should be exactly the portions that went missing in the previous slide.

$$\partial A \cup \partial B = \partial(A \cup B) \cup \partial(A \cap B)$$

Difference

The red ball minus the green ball.

$$\partial(A - B) = \partial(A \cap \overline{B})$$



This is the intersection of the red ball with the complement of the green ball.

The coloring may look a bit odd. You may not expect to see the green at all. It gives much greater flexibility this way. If one prefered, one could attach the red color to the intersection object and achieve an entirely red sphere if desired. But, if that part were to get colored red here by default, there'd be no easy way to make that part green. Oh, and it's far easier to implement it this way.

Quadratics Formulation

$$\vec{x}^T \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{bmatrix} \vec{x}$$

$$+ \vec{x}^T \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} + c \leq 0$$

The advantage of a quadratic surface is that you can use the quadratic formula to calculate the ray surface intersections.

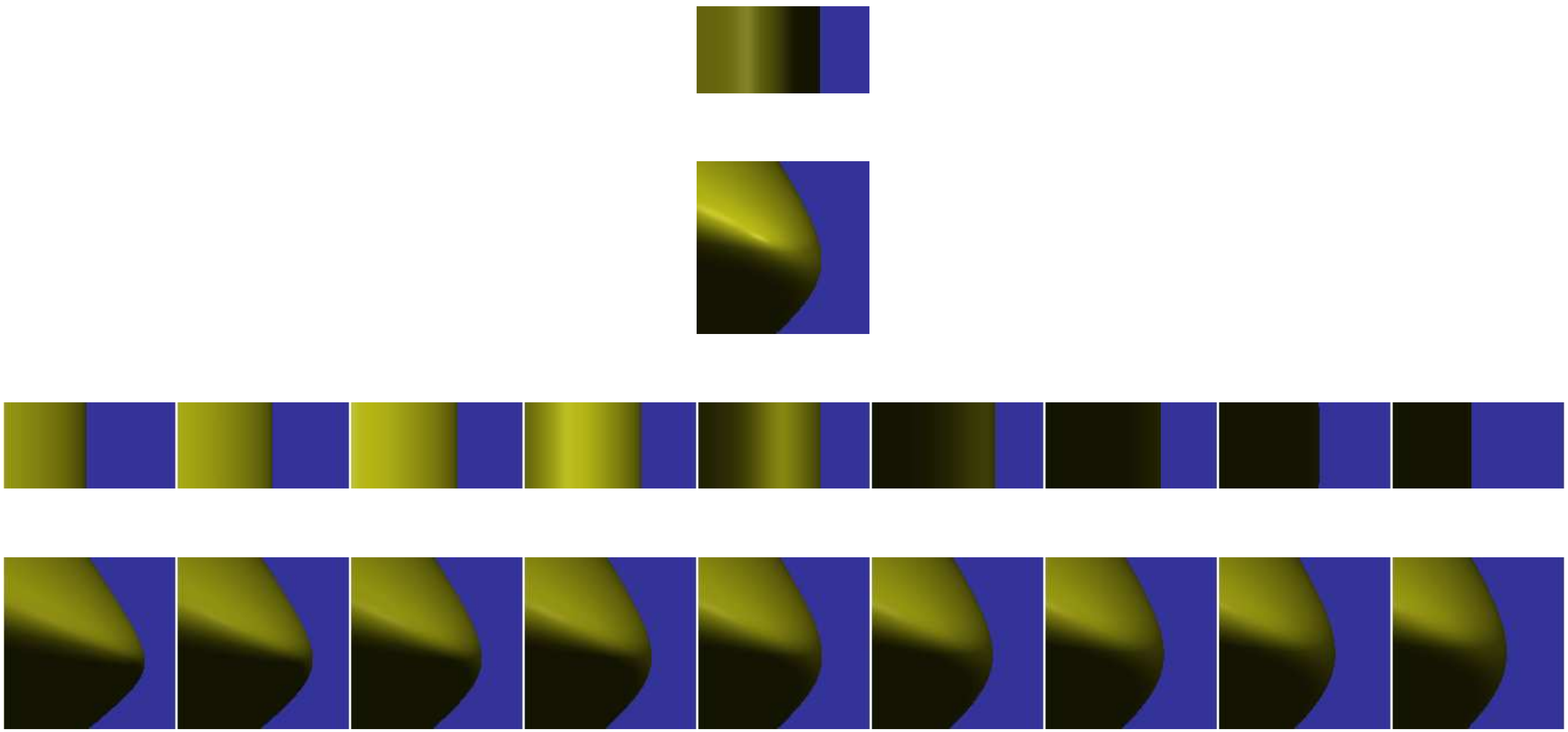
Sorting out the equation above, it may be helpful to think of it as:

$$f(x, y, z, \ldots) = g(x, y, z, \ldots) + h(x, y, z, \ldots) + c$$

where $g(x, y, z, \ldots)$ is a homogeneous polynomial of order 2, $h(x, y, z, \ldots)$ is a homogeneous polynomial of order 1, and c is a constant.

Quadratic Surfaces

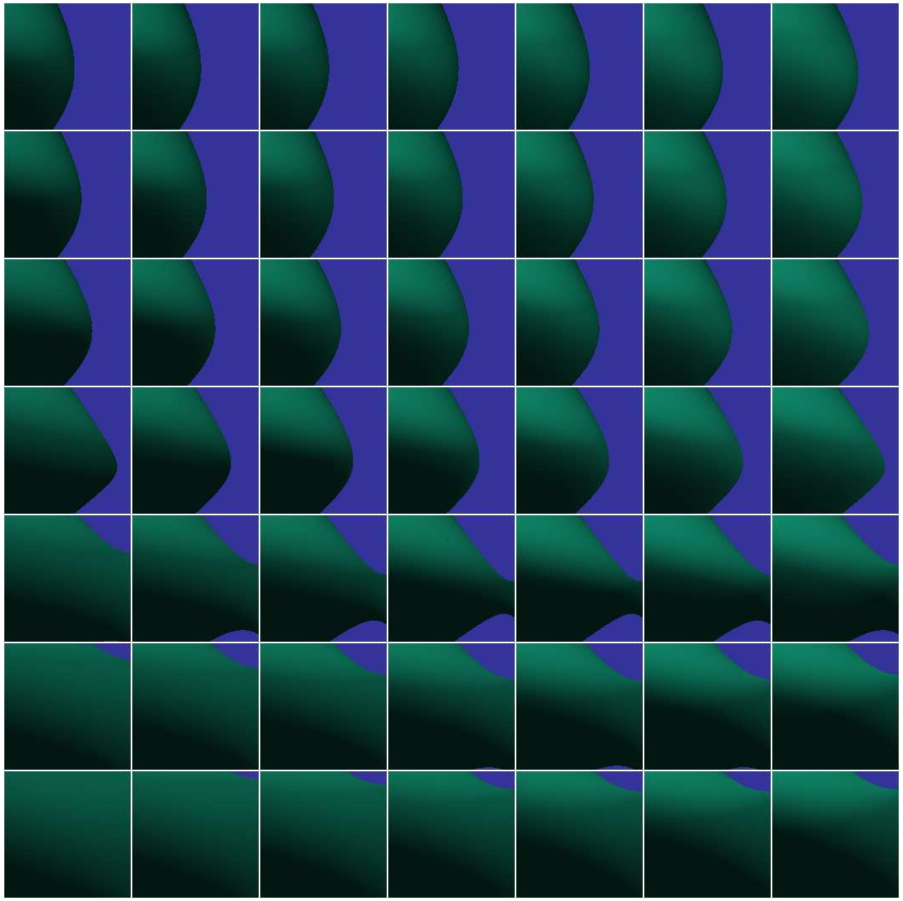
$$-y^2 - 3z^2 + xy + 2yz + x + w - 1 \leq 0$$



For the 2-D picture, all of the z and w are zero.
For the 3-D picture, all of the w are zero.

5-d Quadratic Surface

$$-y^2 - 3z^2 + w^2 + xy + 2yz - wv + x - 5v - 1 \leq 0$$



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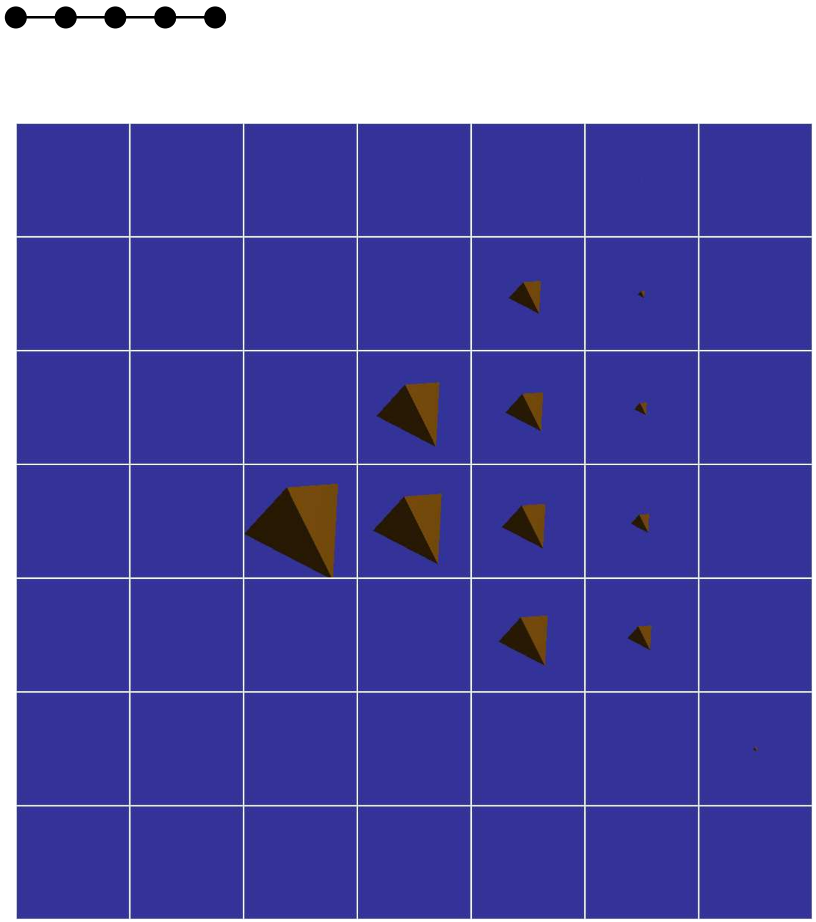
Not much new to say here.

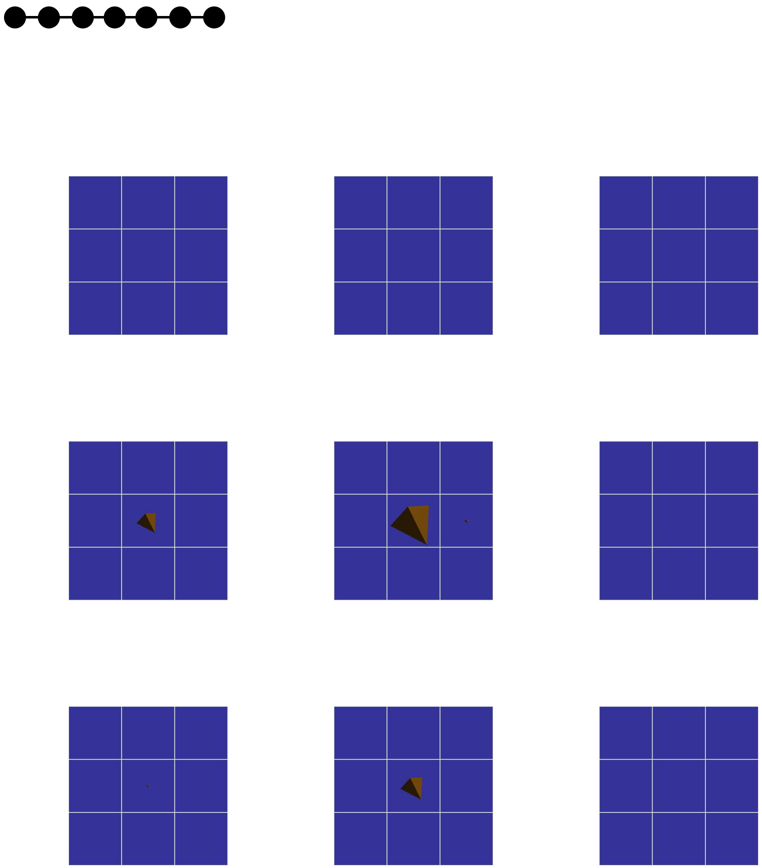
Simplexes

- I'm using Wythoff constructions (thus no Grand Antiprism)
- Vertexes generated by the symmetry group
- Convex hull of vertexes created using intersection of half spaces

The graph beside each picture is the Coxeter-Dynkin diagram for the symmetry group. Each vertex represents a hyperplane of reflection in a fundamental region.

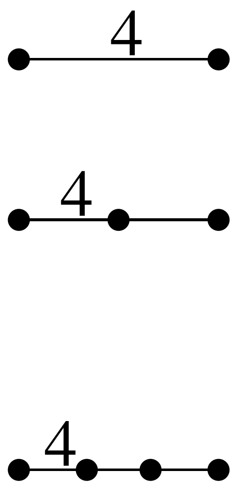
5-d and 7-d Simplexes

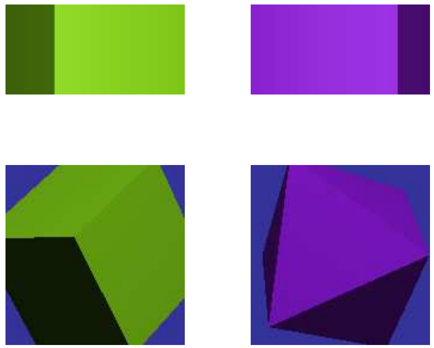


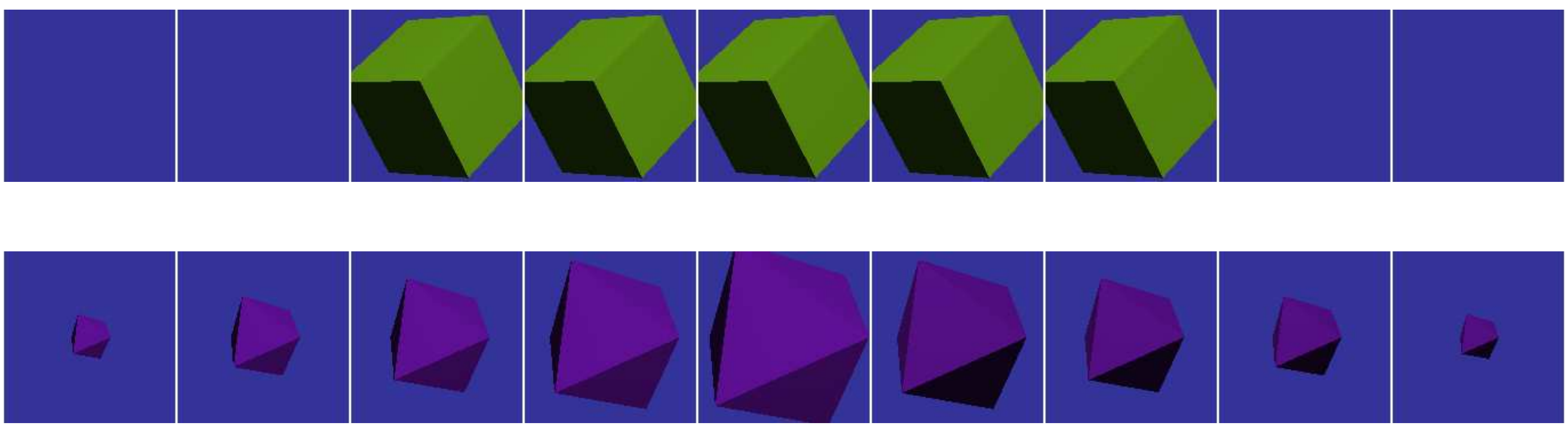


The simplex exists in any number of dimensions.

Cubes and Octahedrons



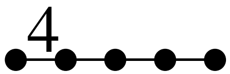


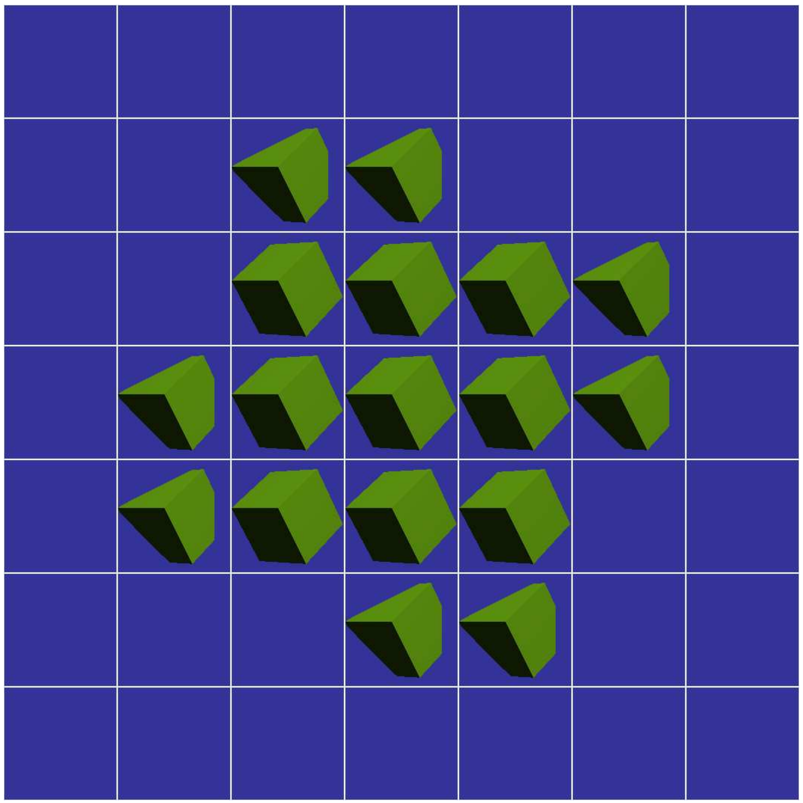


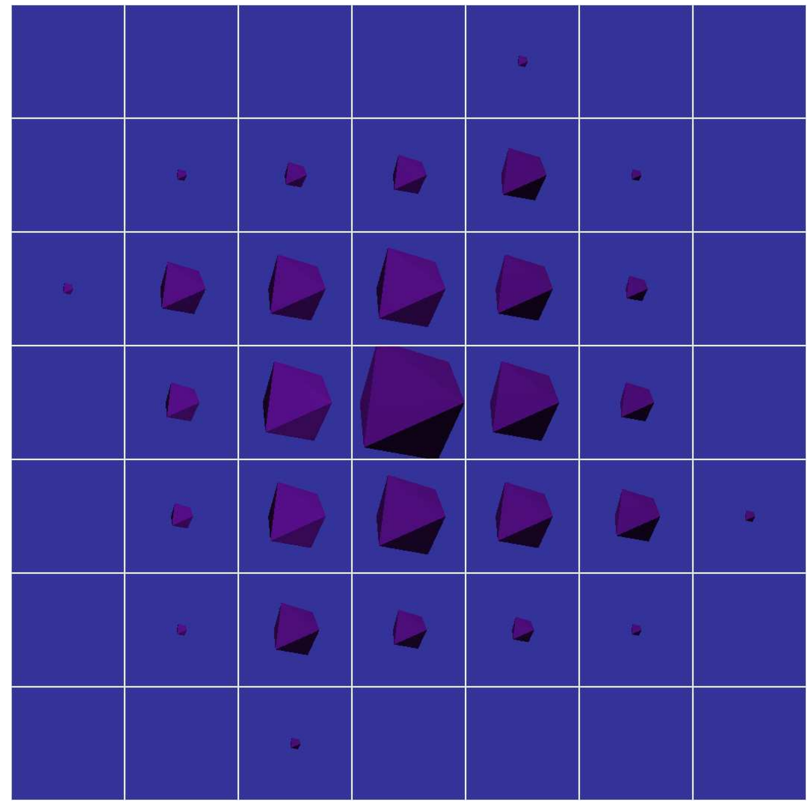
The cube, we saw earlier as a special case of the cylinder. Here, it is generated by its symmetry group.

The cube and the octahedron share the same symmetry group.

5-d Cubes and Octahedrons

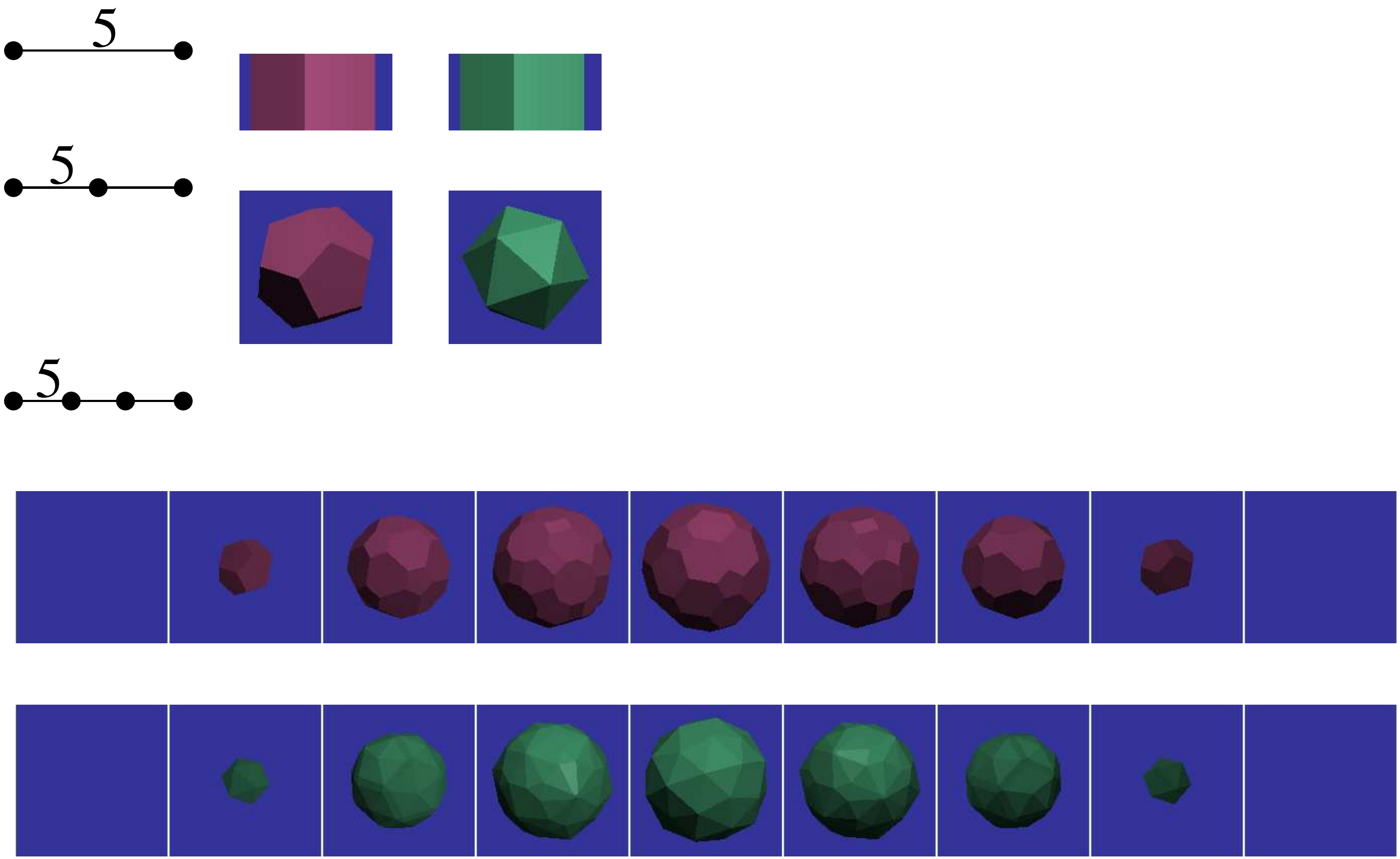






The cube and octahedron both exist in any number of dimensions.

Dodecahedrons and Icosahedrons



The dodecahedron and the icosahedron are dual polyhedron.

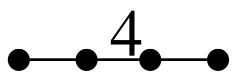
The dodecahedron and the icosahedron are sort of 3-d analogues to the pentagon.

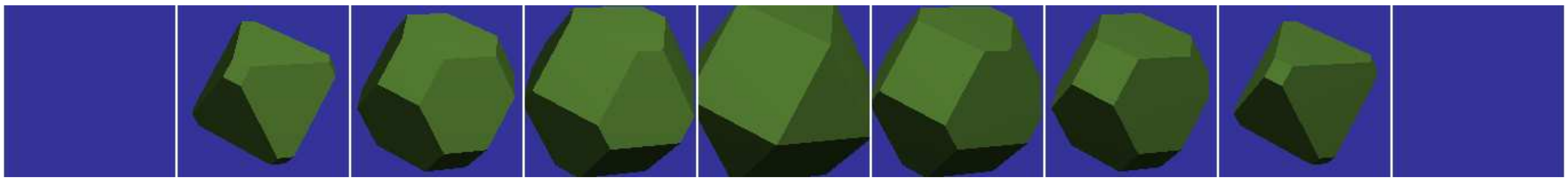
The dodecahedron is the 3-d analogue of the 4-d polytope the 120-cell.

The icosahedron is the 3-d analogue of the 4-d polytope the 600-cell.

The 120-cell and the 600-cell are dual polytopes.

Twenty-Four Cell





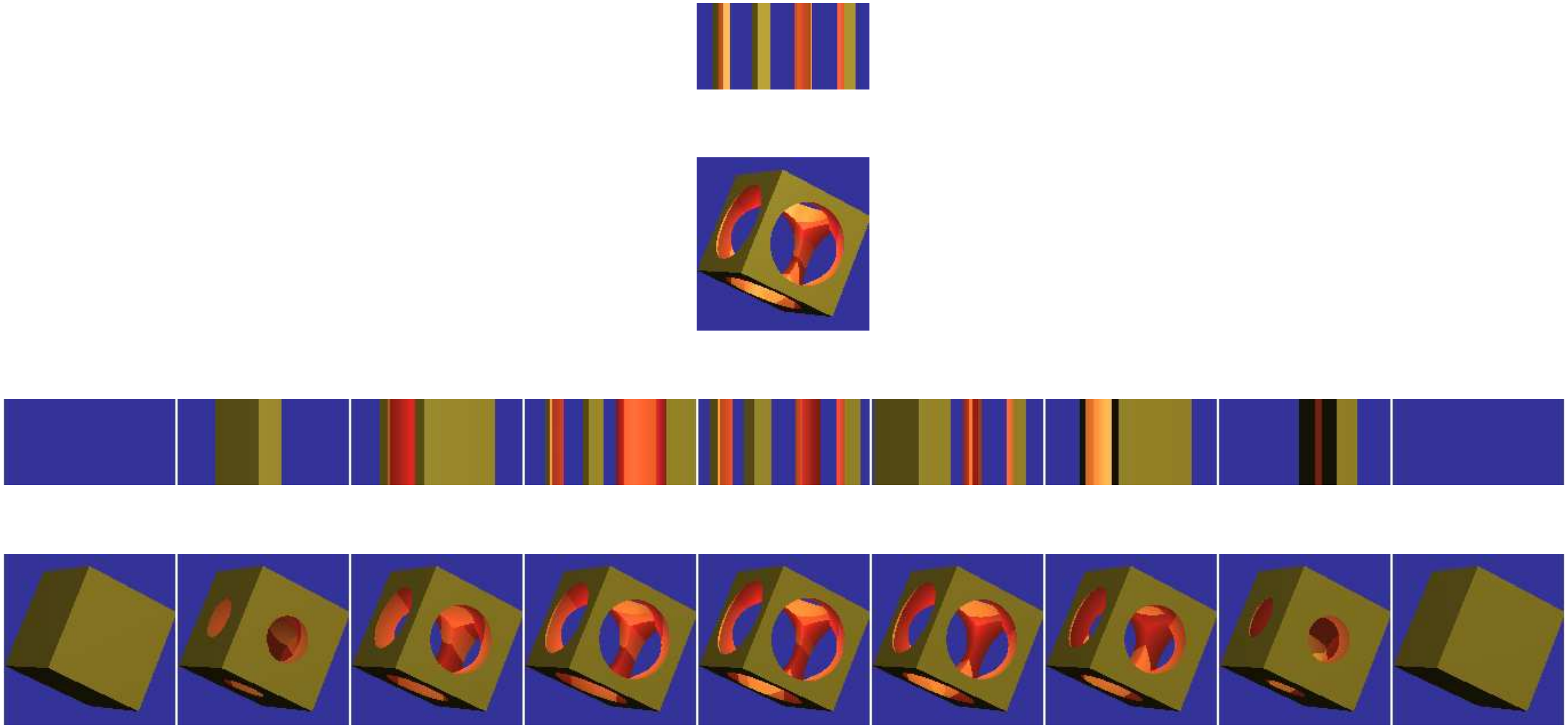
The 24-cell has no direct analogue in 3-d. It is composed of 24-octahedra meeting 3 to an edge.

The 24-cell is self-dual.

But, its intersection with the 3-space where $w = 0$ is the cuboctahedron.

Hole Cube

A cube with a hole drilled through each axis.



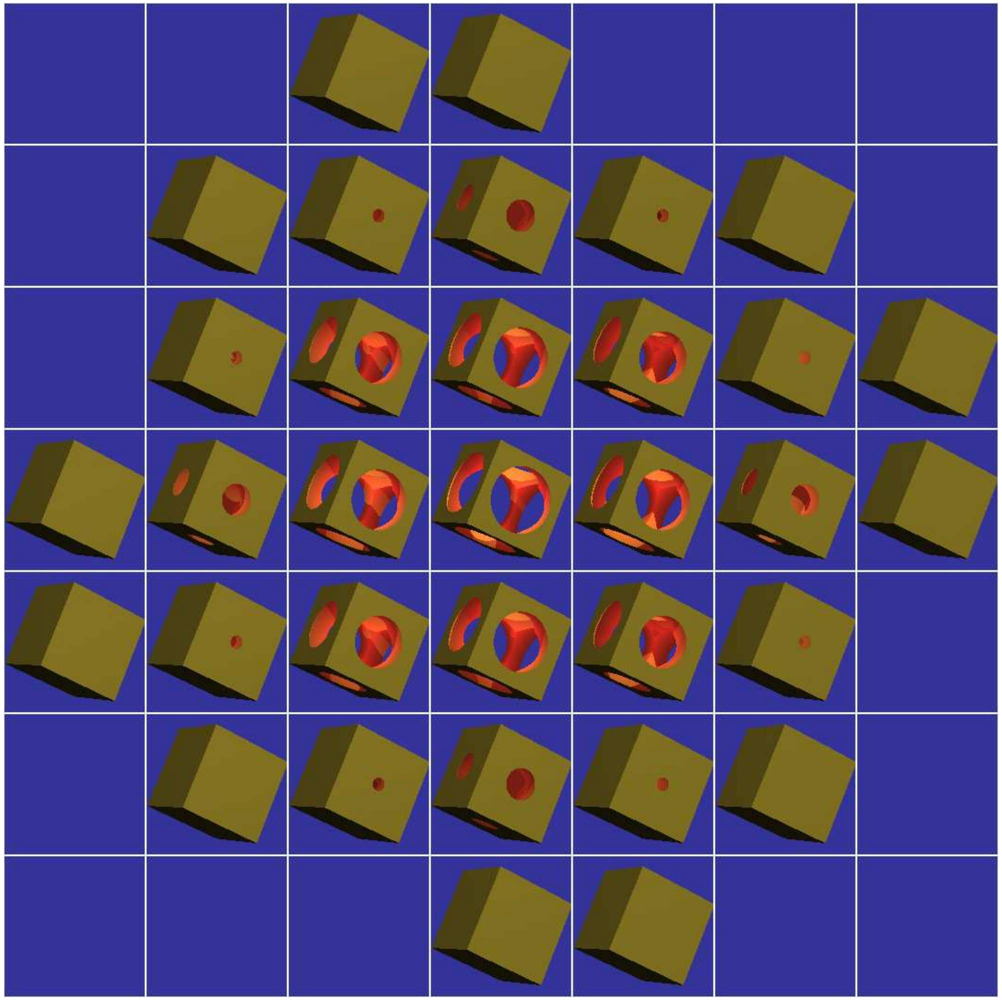
Most of the shapes we've seen so far have been convex.

Things get a fair bit more complicated with shapes passing through lower-dimensional spaces when the shapes are not convex.

If you were in Flatland watching this cube pass through, you'd be hard pressed to know what you saw.

In 3-D, this isn't quite so bad, but it'll be harder to keep track of how things fit together than it was for just the cube.

5-d Hole Cube

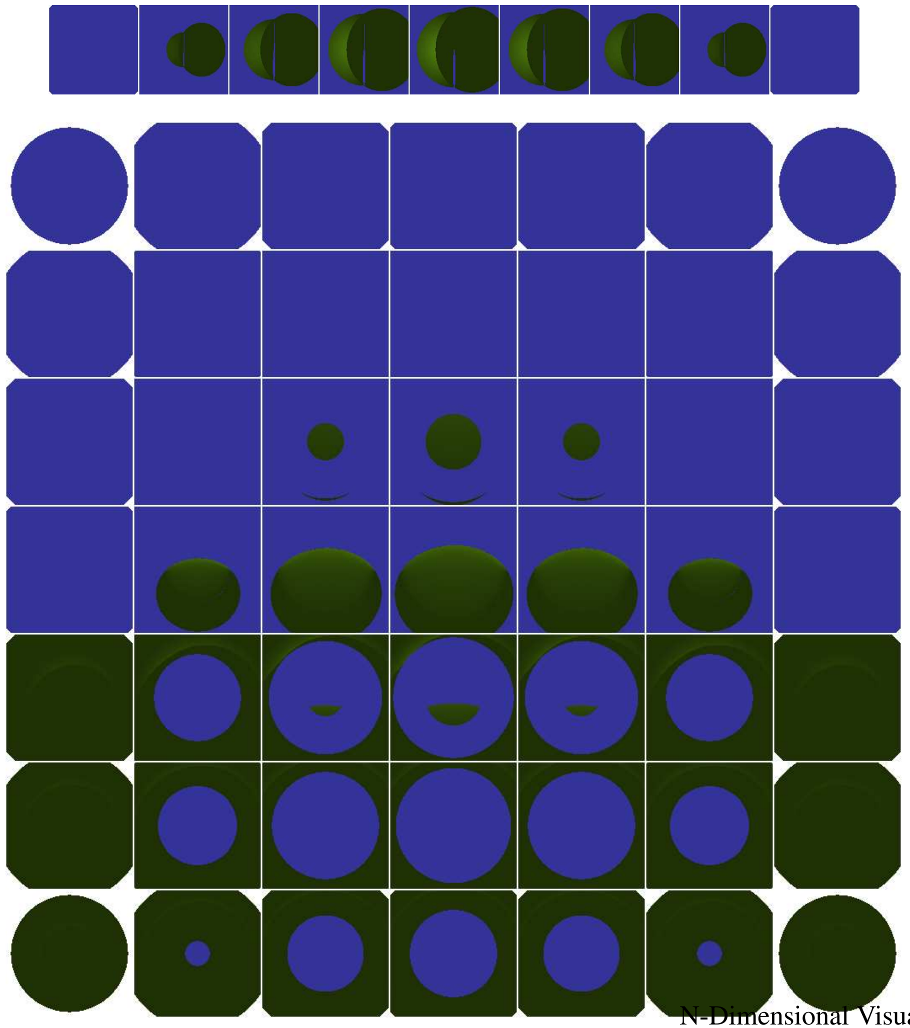


With this 5-d version, you can see that there are some spots where the hole on one face "forms" before the hole on the other faces.

It's interesting to note the squarish shape that the cubes take on in the overall picture as well as how the holes in the front faces form a pattern very similar to that of the 5-d hyperball shown earlier.

Complex Variables

$$f(z) = (z + (1 + 2i))(z + (1 - 2i))$$



Both of these pictures represent the same object. The 5-d version is just oriented so that all four axises of the plot are perpendicular to straight ahead.

Both of these pictures have a massive fish-eye effect to show you much more of the plot.

They are generated as the intersection of two quadratic surfaces.

$$|\Re\{f(z)\}| \leq \frac{1}{2}$$

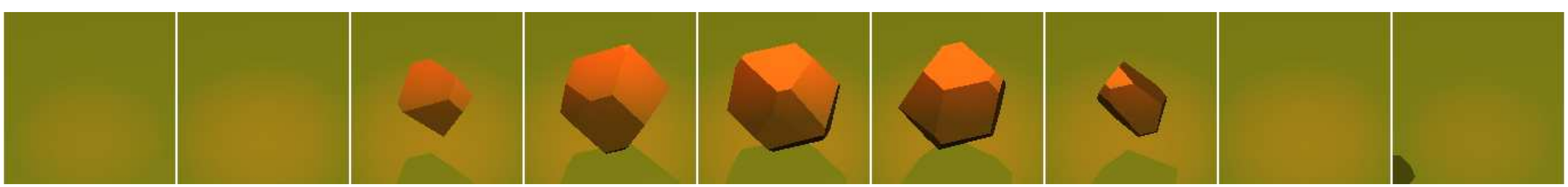
$$|\Im\{f(z)\}| \leq \frac{1}{2}$$

I haven't really studied these enough yet to draw any useful conclusions here.

Convex Hull

Union of a cube and an octahedron

Convex hull of a cube and an octahedron



The cube had vertexes at:

$$\langle \pm 1, \pm 1, \pm 1, \pm 1 \rangle$$

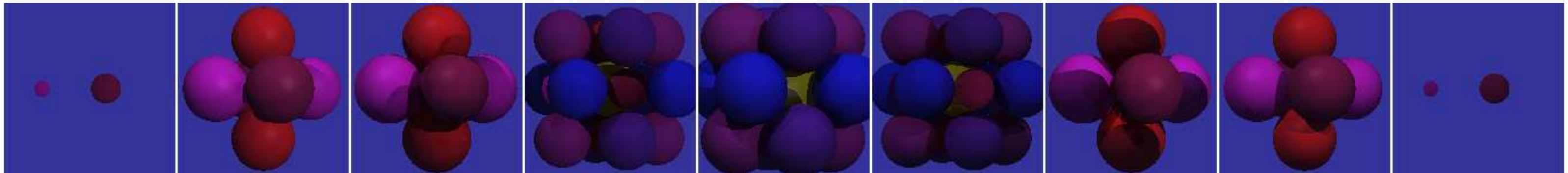
The octahedron had vertexes at permutations of:

$$\langle \pm 2, 0, 0, 0 \rangle$$

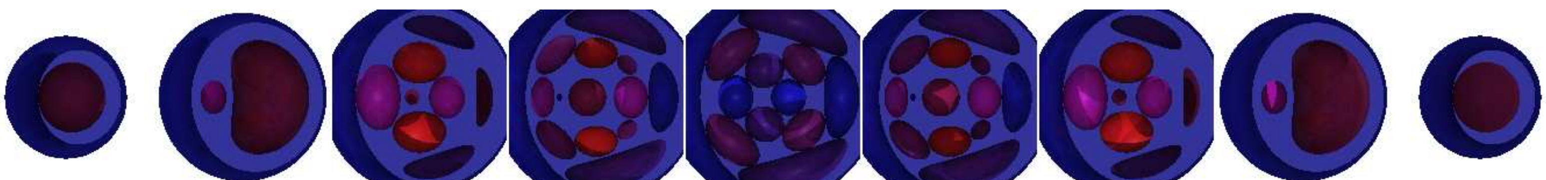
Convex polytopes are at the heart of linear programming.

Convex polytopes often come up in other areas like: base polytopes of a lattice and Vornoi cells.

Kissing Number



Visual validation that the kissing number of S^3 is at least 24.



Same picture viewed fisheyed from inside the center sphere.

We can actually count the 24-spheres. In the center frame at the top, there are 12 spheres visible.

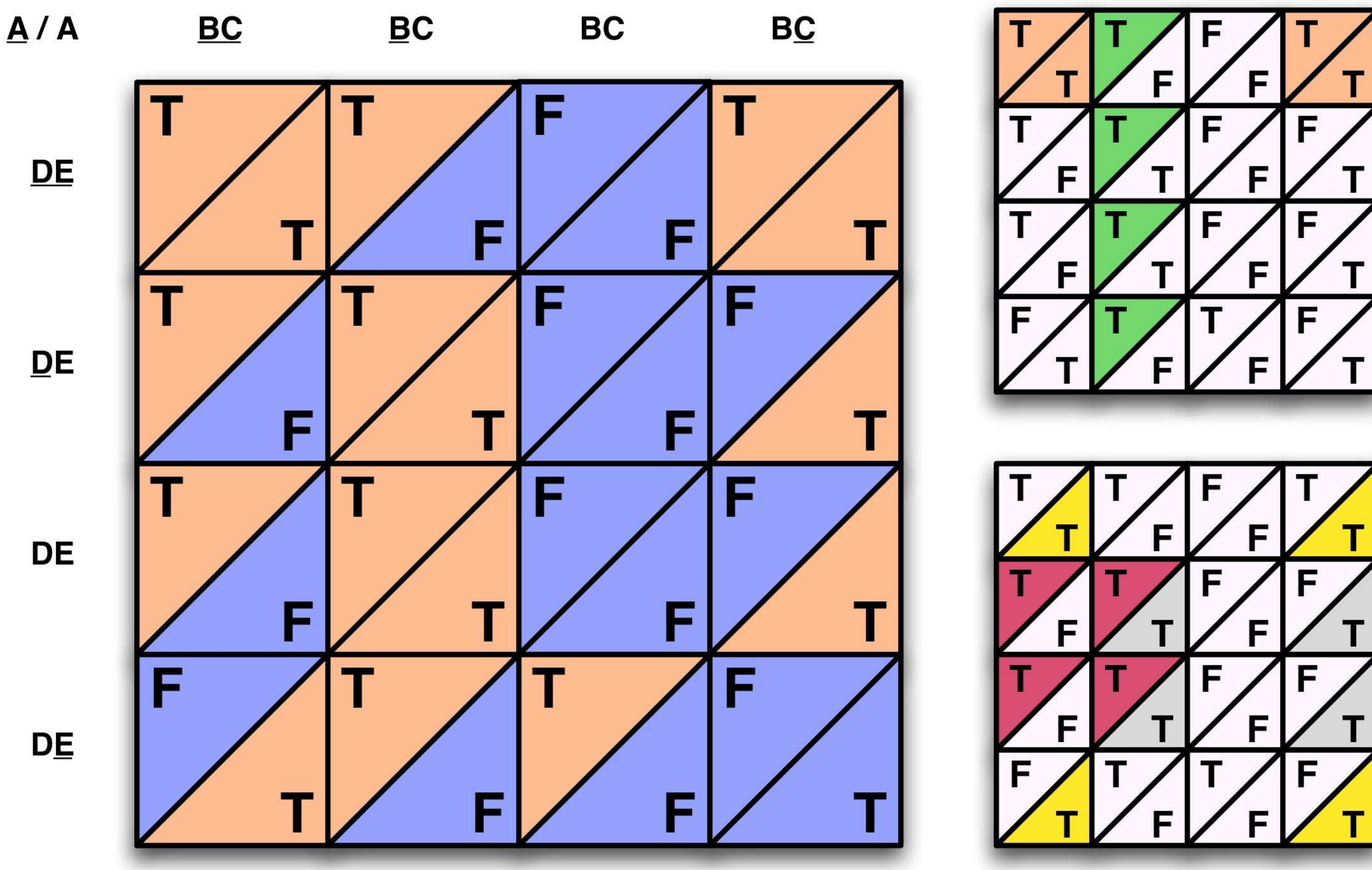
In the third frame, you can see another six. And, in the seventh frame, you can see the final six.

They're harder to count in the bottom sequence. But, they're still there.

The center frame shows 12 spheres.

The dark red ball in the center of frames three and four is opposite the dark red ball in frames one, two, and three. You can see the pink and the bright-red balls in frame three. That's six on this end. There are six on the other end, too.

Karnaugh Maps



Karnaugh maps are a tool used by computer engineers to reduce complex logical expressions down to a sum of products. The goal of that is to be able to implement the function in digital logic in only two steps.

The engineer draws out map like this for his five-variable problem. Then, he circles groupings of trues which are in particular patterns of 1, 2, 4, 8, etc. cells.

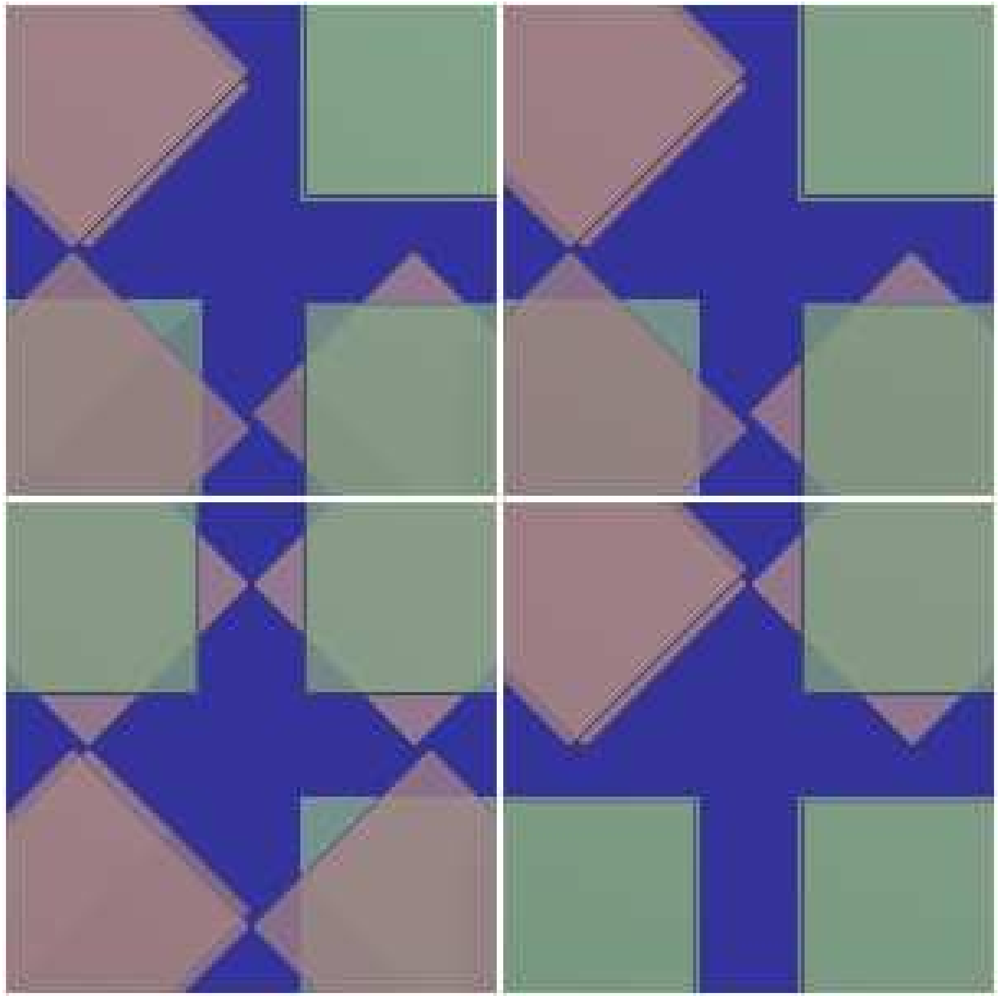
In the pictures at the right, some size 4 cells are highlighted.

The group highlighted in light grey on the bottom picture is not a valid grouping.

This picture is really attempting to show an unfolding of a $2 \times 2 \times 2 \times 2 \times 2$ cube. And, the circled portions are supposed to represent slices through the cube (or through slices already chopped).

So, why not view it that way?

Karnaugh Maps (5-D)



Here, the green squares represent true and the red diamonds represent false.

Some possible groupings here would be to slice the upper frames apart from the lower. Then, slice each upper frame vertically. Then, slice each new right-side piece halfway back. Then, we'd be left with the four green squares on the front-left of the upper frames together.

Credits

- Gödel-Escher-Bach Cube from Amazon.com: `http://www.amazon.com/exec/obidos/tg/detail/-/0465026567`
- Impossible Cube from This Fun's For You: `http://www.thisfunsforyou.com/htdocs/illusions/thecube.php`

Source Code

- Raytracer and input files:
  `http://www.nklein.com/products/rt/`

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